2 MODELING

2.1 Background

Since the 2 DOF Ball Balancer uses two Rotary Servo Base Unit (SRV02) devices and the table is symmetrical, it is assumed that the dynamics of each axis is the same. The 2 DOF Ball Balancer is therefore modeled as two de-coupled "ball and beam" systems where we assume the angle of the x-axis servo only affects the ball movement in the x direction. Similarly for the y ball motion. The equation of motion representing the ball's motion along the x axis relative to the plate angle is developed in Section 2.1.1. The servo angle is introduced into the model in Section 2.1.2 and is then represented as a transfer function in Section 2.1.3.

2.1.1 Nonlinear Equation of Motion

The free body diagram of the Ball and Beam is illustrated in Figure 2.1. Using this diagram, the equation of motion, or *EOM* for short, relating the motion of the ball, x, to the angle of the beam, α, can be found. Based on Newton's First Law of Motion, the sum of forces acting on the ball along the beam equals

$$m_b \ddot{x}(t) = \sum F = F_{x,t} - F_{x,r} \tag{2.1}$$

where m_b is the mass of the ball, x is the ball displacement, $F_{x,r}$ is the force from the ball's inertia, and $F_{x,t}$ is the translational force generated by gravity. Friction and viscous damping are neglected.

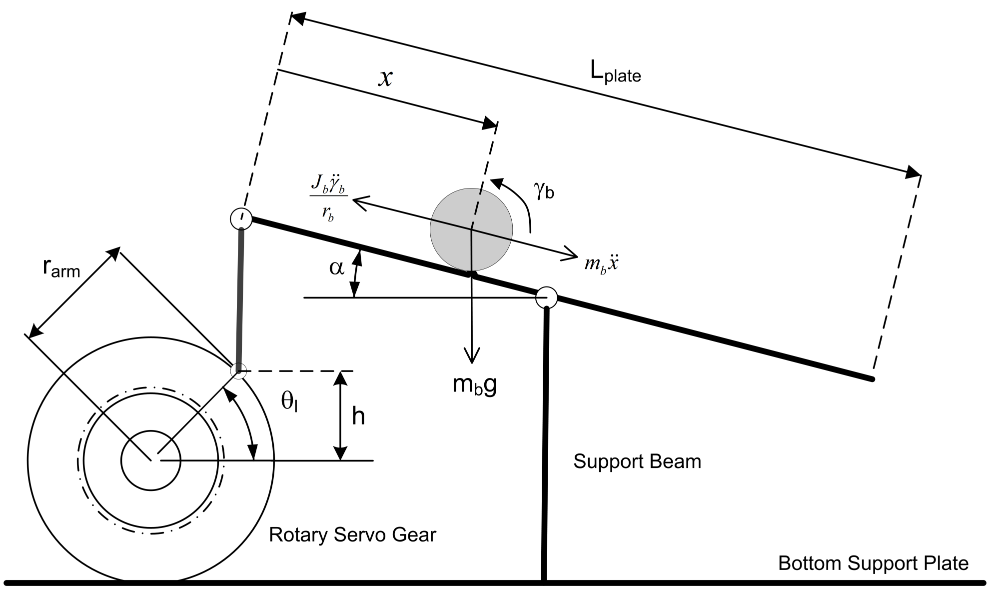

Figure 2.1: Modeling ball on plate in one dimension.

Modeling Conventions:

- Applying a positive voltage causes the servo load gear to move in the positive, counter-clockwise (CCW) direction. This moves the beam upwards and causes the ball to roll in the positive direction (i.e., away from the servo towards the left). Thus $V_m > 0 \rightarrow \dot{\theta}_l > 0 \rightarrow \dot{x} > 0$.
- Ball position is zero, $x = 0$, when located in the center of the beam.
- Servo angle is zero, $\theta_l = 0$, when the beam is parallel to the ground, $\alpha = 0$.

For the ball to be stationary at a certain moment, i.e., be in equilibrium, the force from the ball's momentum must be equal to the force produced by gravity. As illustrated in Figure 2.1, the force $F_{x,t}$ in the x direction (along the beam) that is caused by gravity can be found as:

$$F_{x,t} = m_b\, g\, \sin\alpha(t)$$

The force caused by the rotation of the ball is

$$F_{x,r} = \frac{\tau_b}{r_b}$$

where r_b is the radius of the ball and τ_b is the torque which equals

$$\tau_b = J_b \ddot{\gamma}_b(t)$$

where γ_b is the ball angle. Using the sector formula, $x(t) = \gamma_b(t)\, r_b$, we can convert between linear and angular displacement. Then, the force acting on the ball in the x direction from its momentum becomes:

$$F_{x,r} = \frac{J_b \ddot{x}(t)}{r_b^2}.$$

Now, by substituting the rotational and translational forces into Equation 2.1, we can get the nonlinear equation of motion for the ball and beam as:

$$m_b\, \ddot{x}(t) = m_b\, g\, \sin\alpha(t) - \frac{J_b\, \ddot{x}(t)}{r_b^2}.$$

Solving for the linear acceleration gives:

$$\ddot{x}(t) = \frac{m_b\, g\, \sin\alpha(t)\, r_b^2}{m_b\, r_b^2 + J_b}. \tag{2.2}$$

2.1.2 Relative to Servo Angle

In this section, the equation of motion representing the position of the ball relative to the angle of the servo load gear is derived. The obtained equation will be nonlinear (includes a trigonometric term). Therefore, it will have to be linearized to use in control design.

Let's look at how we can find the relationship between the servo load gear angle, θ_l, and the beam angle, α. Using the schematic given in Figure 2.1, consider the beam and servo angles required to change the height of the beam by h. Taking the sine of the beam angle gives the expression

$$\sin\alpha(t) = \frac{2h}{L_{plate}}$$

and taking the sine of the servo load shaft angle results in the equation

$$\sin\theta_l(t) = \frac{h}{r_{arm}}.$$

From these we can obtain the following relationship between the beam and servo angle

$$\sin\alpha(t) = \frac{2\, r_{arm}\, \sin\theta_l(t)}{L_{plate}}. \tag{2.3}$$

To find the equation of motion that represent the ball's motion with respect to the servo angle θ_l we need to linearize the equation of motion about the servo angle $\theta_l(t) = 0$. Insert the servo and plate angle relationship, Equation 2.3, into the nonlinear EOM found in 2.2

$$\ddot{x}(t) = \frac{2\, m_b\, g\, r_{arm}\, r_b^2}{L_{plate}\, (m_b\, r_b^2 + J_b)}\, \sin\theta_l(t). \tag{2.4}$$

About angle zero, the sine function can be approximated by $\sin\theta_l(t) \approx \theta_l(t)$. Applying this to the nonlinear EOM gives the *linear* equation of motion of the ball

$$\ddot{x}(t) = \frac{2\, m_b\, g\, r_{arm}\, r_b^2}{L_{plate}\, (m_b\, r_b^2 + J_b)}\, \theta_l(t). \tag{2.5}$$

2.1.3 Obtaining the Transfer Function

The complete open-loop system of the 2 DOF Ball Balancer is represented by the block diagram shown in Figure 2.2. The Rotary Servo Base Unit (SRV02) transfer function $P_s(s)$ represents the dynamics between the servo input motor voltage and the resulting load angle. The dynamics between the angle of the servo load gear and the position of the ball is described by transfer function $P_{bb}(s)$. This is a decoupled model, e.g., it is assumed the x-axis servo does not affect the y-axis response.

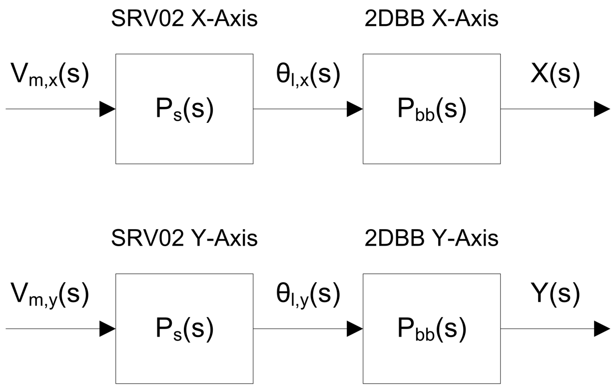

Figure 2.2: 2 DOF Ball Balancer open-loop block diagram

The block diagram for a single-axis of the 2 DOF Ball Balancer, denoted 1DBB, is shown in Figure 2.3.

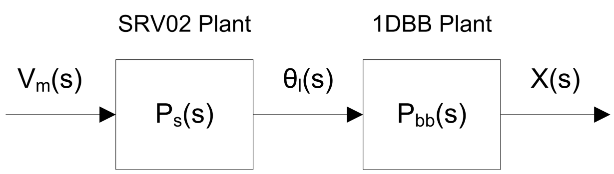

Figure 2.3: 1D open-loop block diagram of 2 DOF Ball Balancer

This section will describe how to obtain the 1DBB transfer function

$$P(s) = P_{bb}(s)\, P_s(s)$$

where

$$P_{bb}(s) = \frac{X(s)}{\Theta_l(s)}$$

is the *servo angle to ball position* transfer function and

$$P_s(s) = \frac{\Theta_l(s)}{V_m(s)}$$

is the *voltage to servo angle* transfer function.

The SRV02 transfer function models the servo load gear position, $\theta_l(t)$, with respect to the servo input voltage, $V_m(t)$. Recall that in Modeling Laboratory ([5]), this transfer function was found to be:

$$P_s(s) = \frac{K}{s\,(\tau s + 1)}. \tag{2.6}$$

The nominal model parameters, K and τ, of the SRV02 with no load and in high-gear configuration are:

$$K = 1.53 \text{ rad/(V-s)} \tag{2.7}$$

and

$$\tau = 0.0248 \text{ s} \tag{2.8}$$

Note: These parameters are different than the those found in the Modeling Laboratory because it does not include the inertial disc load.

The servo angle to ball position transfer function, $P_{bb}(s)$, can be found by taking the Laplace transform of the linear equation of motion in 2.5 as:

$$P_{bb}(s) = \frac{X(s)}{\Theta_l(s)} = \frac{K_{bb}}{s^2} \tag{2.9}$$

As illustrated in Figure 2.3, both systems are in series. By inserting the plate position transfer function, 2.9, and the voltage-servo transfer function, 2.6, into Equation 2.10, we can derive the complete process transfer function $P(s)$ as:

$$P(s) = \frac{X(s)}{V_m(s)} = \frac{K_{bb}\,K}{s^3\,(\tau\,s + 1)} \tag{2.10}$$

This is the servo voltage to ball displacement transfer function.